Κατεύθυνση Β' Λυκείου, Λύσεις

Θέμα Α

Α1 $\lim_{x\to -1^-} \frac{x^2+4x+3}{x^2-1} = \lim_{x\to -1^-} \frac{(x+1)(x+3)}{(x-1)(x+1)} = -1$

$\lim_{x\to -1^+} (\kappa x+\mu) = -\kappa+\mu$

Άρα $-\kappa+\mu=-1$ (1)

$\lim_{x\to 1^-} (\kappa x+\mu) = \kappa+\mu$

$\lim_{x\to 1^+} \frac{x-\sqrt{2-x}}{x-\sqrt{x}} = \lim_{x\to 1^+} \frac{x^2+x-2}{(x-\sqrt{x})(x+\sqrt{2-x})} = \lim_{x\to 1^+} \frac{(x+2)(x-1)}{\sqrt{x}(\sqrt{x}-1)(x+\sqrt{2-x})} =$

$= \lim_{x\to 1^+} \frac{(x+2)\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}{\sqrt{x}\cancel{(\sqrt{x}-1)}(x+\sqrt{2-x})} = \frac{3\cdot 2}{2} = 3$

Άρα $\kappa+\mu=3$ (2)

$\begin{cases} -\kappa+\mu=-1 \\ \kappa+\mu=3 \end{cases}$ | $2\mu=2 \Leftrightarrow \boxed{\mu=1 \\ \kappa=2}$

Α2 (1) Έστω $g(x)=\frac{(x^2-1)f(x)+\sqrt{x+3}-2}{x-1}$ με $\lim_{x\to 1} g(x)=\frac{25}{4}$

$\overset{x\neq 1}{\Leftrightarrow} (x-1)f(x)+\sqrt{x+3}-2=(x-1)g(x) \Leftrightarrow$

$\Leftrightarrow f(x)=\frac{(x-1)g(x)+2-\sqrt{x+3}}{x^2-1}$. Τότε

$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left[\frac{g(x)}{x+1} - \frac{\sqrt{x+3}-2}{x^2-1}\right] =$

$$= \lim_{x\to 1}\left[\frac{g(x)}{x+1} - \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3}+2)(x+1)}\right] = \frac{25}{8} - \frac{1}{8} = \frac{24}{8} = 3$$

(ii) $\lim_{x\to 1} \dfrac{f^2(x)-3f(x)}{\sqrt{f(x)+1}-2} = \lim_{x\to 1} \dfrac{f(x)\left[\cancel{f(x)-3}\right]\left(\sqrt{f(x)+1}+2\right)}{\cancel{f(x)-3}} = 12.$

A3 Έστω $u = \sqrt[6]{x+3}$. Τότε $u_0 = \lim_{x\to -2} \sqrt[6]{x+3} = 1$

Επίσης $u^2 = \sqrt[3]{x+3}$, $u^3 = \sqrt{x+3}$ και $u^6 = x+3 \Leftrightarrow x = u^6 - 3$.

Οπότε έχουμε $\lim_{u\to 1} \dfrac{u^3+u^6-2}{u^3+u^2-2} = \lim_{u\to 1} \dfrac{(u^3+2)(u^3-1)}{u^3+u^2-2} =$

$$= \lim_{u\to 1} \frac{(u^3+2)\cancel{(u-1)}(u^2+u+1)}{\cancel{(u-1)}(u^2+2u+2)} = \frac{9}{5}$$

B1 $f^2(x) - 2xf(x) \leq \ln^2 x - 2x\ln x \Leftrightarrow$

$\Leftrightarrow f^2(x) - 2xf(x) + x^2 \leq \ln^2 x - 2x\ln x + x^2 \Leftrightarrow$

$\Leftrightarrow (f(x)-x)^2 \leq (\ln x - x)^2$

$\Leftrightarrow |f(x)-x| \leq \sqrt{(\ln x - x)^2} \Leftrightarrow$

$\Leftrightarrow -|\ln x - x| \leq f(x) - x \leq |\ln x - x| \Leftrightarrow$

$\Leftrightarrow x - |\ln x - x| \leq f(x) \leq x + |\ln x - x|$

$$\left.\begin{aligned} &\lim_{x\to 0}(x - |\ln x - x|) = 0 \\ &\lim_{x\to 0}(x + |\ln x - x|) = 0 \end{aligned}\right\} \text{Από ΚΠ. } \lim_{x\to 0} f(x) = 0$$

B2 Ισχυει οτι $\lim_{x\to 2} f(x+3) = 4 \overset{u=x+3}{\underset{\substack{x\to 2\\ u\to 5}}{\Longleftrightarrow}} \lim_{u\to 5} f(u) = 4$

Τοτε $\lim_{x\to -5} f(x) \overset{y=-x}{\underset{\substack{x\to -5\\ y\to 5}}{=}} \lim_{y\to 5} f(-y) \overset{f \text{ περιττη}}{=} -\lim_{y\to 5} f(y) = -4$

B3 Αν $4\sqrt{x} \leq f(x) \leq x+4$ (1) τοτε

(i) $\lim_{x\to 4} \quad = 8$

$\lim_{x\to 4} (x+4) = 8$

$\left.\begin{matrix} \lim_{x\to 4} \quad = 8 \\ \lim_{x\to 4}(x+4)=8 \end{matrix}\right\}$ Απο ΚΠ $\lim_{x\to 4} f(x) = 8$

(ii) (1) $\Longleftrightarrow 4\sqrt{x} - 8 \leq f(x) - 8 \leq x-4$

$\boxed{\text{Για } x<4}$

$$\frac{4(\sqrt{x}-2)}{x-4} \geq \frac{f(x)-8}{x-4} \geq \frac{x-4}{x-4}$$

$\left.\begin{matrix} \lim_{x\to 4^-} \dfrac{4(\cancel{\sqrt{x}-2})}{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)} = 1 \\ \lim_{x\to 4^-} \dfrac{x-4}{x-4} = 1 \end{matrix}\right\}$ Απο ΚΠ $\lim_{x\to 4^-} \dfrac{f(x)-8}{x-4} = 1$

$\boxed{\text{Για } x>4}$

$$\frac{4(\sqrt{x}-2)}{x-4} \leq \frac{f(x)-8}{x-4} \leq \frac{x-4}{x-4}$$

$\left.\begin{matrix} \lim_{x\to 4^+} \dfrac{4(\sqrt{x}-2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = 1 \\ \lim_{x\to 4^+} \dfrac{x-4}{x-4} = 1 \end{matrix}\right\}$ Απο Κ.Π $\lim_{x\to 4^+} \dfrac{f(x)-8}{x-4} = 1$

Αρα $\lim_{x\to 4} \dfrac{f(x)-8}{x-4} = 1$

(iii) $\lim\limits_{x\to 4} \dfrac{f^2(x)-64}{x-4} = \lim\limits_{x\to 4}\left[\dfrac{f(x)-8}{x-4}(f(x)+8)\right] = 16$

(iv) $\lim\limits_{x\to 4} \dfrac{\sqrt{f(x)+1}-3}{x-4} = \lim\limits_{x\to 4}\left[\dfrac{f(x)-8}{x-4}\cdot\dfrac{1}{\sqrt{f(x)+1}+3}\right] = \dfrac{1}{6}$

(v) $\lim\limits_{x\to 4} |f(x)-5| = 3 > 0$ αρα $f(x)-5 > 0$ κοντα στο 4.

Αρα $|f(x)-5| = f(x)-5$

$\lim\limits_{x\to 4} \dfrac{f(x)-5-3}{x^2-5x+4} = \lim\limits_{x\to 4} \dfrac{f(x)-8}{(x-4)(x-1)} = \dfrac{1}{3}$

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Μέρος Β': Άλγεβρα

ΘΕΜΑ Γ

Γ1 Σχολικό βιβλίο → σελ. 191

Γ2
1) Σωστό
2) Λάθος
3) Σωστό
4) Λάθος
5) Λάθος

Γ3 i) Έχουμε: $\frac{x^3}{4} + \frac{x^2}{2} - 5x + 6 = 0 \overset{\cdot 4}{\Leftrightarrow} x^3 + 2x^2 - 20x + 24 = 0$

$\Leftrightarrow (x-2)(x^2+4x-12) = 0$

$\Leftrightarrow (x-2)\cdot(x-2)(x+6) = 0$

$\Leftrightarrow (x-2)^2\cdot(x+6) = 0$

$\Leftrightarrow \boxed{x=2}$ ή $\boxed{x=-6}$

1	2	−20	24	2
	2	8	−24	
1	4	−12	0	

ii) Έχουμε: $\sqrt{x+7} + \sqrt{6-x} = 5$ (1)

Πρέπει: $x+7 \geq 0 \Leftrightarrow x \geq -7$
$6 - x \geq 0 \Leftrightarrow x \leq 6$ $\Big\} \Rightarrow -7 \leq x \leq 6$

Από (1) $\Leftrightarrow \left(\sqrt{x+7} + \sqrt{6-x}\right)^2 = 5^2$

$\Leftrightarrow \sqrt{x+7}^2 + 2\sqrt{x+7}\cdot\sqrt{6-x} + \sqrt{6-x}^2 = 25$

$\Leftrightarrow \cancel{x} + 7 + 6 - \cancel{x} + 2\cdot\sqrt{(x+7)(6-x)} = 25$

$\Leftrightarrow 2\cdot\sqrt{6x - x^2 + 42 - 7x} = 12$

$\Leftrightarrow \sqrt{-x^2 - x + 42} = 6$

$\Leftrightarrow -x^2 - x + 42 = 36$

$\Leftrightarrow x^2 + x - 6 = 0$

$\Leftrightarrow \boxed{x=-3}$ ή $\boxed{x=2}$

δεκτή δεκτή

Γ4 i) Έχουμε: $\dfrac{3x^2-6x+2}{x-2} - \dfrac{2}{(x-1)(x-2)} - \dfrac{x^2-5x+6}{x-1} \geq 0$ $\quad x \neq 1,\ x \neq 2$

$$\Leftrightarrow \frac{(x-1)(3x^2-6x+2)-2-(x-2)(x^2-5x+6)}{(x-1)(x-2)} \geq 0$$

$$\Leftrightarrow \frac{3x^3-6x^2+2x-3x^2\cancel{+6x}-2-2-x^3+5x^2\cancel{-6x}+2x^2-10x+12}{(x-1)(x-2)} \geq 0$$

$$\Leftrightarrow \frac{2x^3-2x^2-8x+8}{(x-1)(x-2)} \geq 0$$

$$\Leftrightarrow 2\cdot(x^3-x^2-4x+4)(x-1)(x-2) \geq 0$$

1	-1	-4	4	1
	1	0	-4	
1	0	-4	0	

$$\Leftrightarrow (x-1)(x^2-4)(x-1)(x-2) \geq 0$$

$$\Leftrightarrow (x-1)(x-2)(x+2)(x-1)(x-2) \geq 0$$

$$\Leftrightarrow (x-1)^2(x-2)^2(x+2) \geq 0$$

$$\Leftrightarrow x+2 \geq 0$$

$$\Leftrightarrow x \geq -2$$

και $x \neq 1,\ x \neq 2$ $\Big\}\Rightarrow$ $\boxed{x \in [-2,1)\cup(1,2)\cup(2,+\infty)}$

ii) Έχουμε $\sqrt{2x+1} \leq \sqrt{x+3}$ (1)

πρέπει: $2x+1 \geq 0 \Leftrightarrow x \geq -\frac{1}{2}$
$x+3 \geq 0 \Leftrightarrow x \geq -3$ $\Big\}\Rightarrow x \geq -\frac{1}{2}$

Από (1) $\Leftrightarrow \sqrt{2x+1}^{\,2} \leq \sqrt{x+3}^{\,2}$

$\Leftrightarrow 2x+1 \leq x+3$

$\Leftrightarrow x \leq 2$

άρα $\boxed{x \in \left[-\frac{1}{2}, 2\right]}$

ΘΕΜΑ Δ

Έχουμε: $f(x) = x^4 - 8x^3 + (5a-1)x^2 + 8x - 3a - \beta$

και δίνεται: $f(-1) = 0$ και $f(-2) = 105$

Δ1 $\left.\begin{array}{l} f(-1)=0 \\ f(-2)=105 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} (-1)^4 - 8(-1)^3 + (5a-1)\cdot(-1)^2 + 8\cdot(-1) - 3a - \beta = 0 \\ (-2)^4 - 8(-2)^3 + (5a-1)\cdot(-2)^2 + 8(-2) - 3a - \beta = 105 \end{array}\right\} \Leftrightarrow$

$\left.\begin{array}{l} 1 + 8 + 5a - 1 - 8 - 3a - \beta = 0 \\ 16 + 64 + 20a - 4 - 16 - 3a - \beta = 105 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} 2a - \beta = 0 \\ 17a - \beta = 45 \end{array}\right\} \Leftrightarrow$

$\left.\begin{array}{l} \beta = 2a \\ 17a - 2a = 45 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} \beta = 2a \\ 15a = 45 \end{array}\right\} \Leftrightarrow \begin{array}{l} \boxed{\beta = 6} \\ \boxed{a = 3} \end{array}$

Άρα: $f(x) = x^4 - 8x^3 + 14x^2 + 8x - 15$, $x \in \mathbb{R}$

Δ2 Είναι $f(x) = 0 \Leftrightarrow x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$

$\Leftrightarrow (x-1)(x^3 - 7x^2 + 7x + 15) = 0$

$\Leftrightarrow (x-1)(x+1)(x^2 - 8x + 15) = 0$

$\Leftrightarrow x - 1 = 0$ ή $x + 1 = 0$ ή $x^2 - 8x + 15 = 0$

$\Leftrightarrow \boxed{x = 1}$ $\boxed{x = -1}$ $\boxed{x = 3}$ ή $\boxed{x = 5}$

1	-8	14	8	-15	1
	1	-7	7	15	
1	-7	7	15	0	

1	-7	7	15	-1
	-1	8	-15	
1	-8	15	0	

Δ3 Είναι: $f(x) < 0 \Leftrightarrow (x-1)(x+1)(x^2 - 8x + 15) < 0$

x	$-\infty$	-1		1		3		5	$+\infty$
$x-1$	$-$		$-$	0	$+$		$+$		$+$
$x+1$	$-$	0	$+$		$+$		$+$		$+$
$x^2-8x+15$	$+$		$+$		$+$	0	$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$	0	$-$	0	$+$

άρα $\boxed{x \in (-1, 1) \cup (3, 5)}$

Δ4 • Αφού $-\sqrt{3} < -1$, τότε $f(-\sqrt{3}) > 0$

• Αφού $-1 < \frac{1821}{2021} < 1$, τότε $f\left(\frac{1821}{2021}\right) < 0$

• Αφού $1 < \sqrt{5} < 3$, τότε $f(\sqrt{5}) > 0$

• Είναι $\frac{10}{\sqrt{10}} = \frac{10 \cdot \sqrt{10}}{10} = \sqrt{10}$

Αφού $3 < \sqrt{10} < 5$, τότε $f(\sqrt{10}) < 0$

• Αφού $28 > 5$, τότε $f(28) > 0$

Άρα $\Pi = \underbrace{f(-\sqrt{3})}_{(+)} \cdot \underbrace{f\left(\frac{1821}{2021}\right)}_{(-)} \cdot \underbrace{f(\sqrt{5})}_{(+)} \cdot \underbrace{f(\sqrt{10})}_{(-)} \cdot \underbrace{f(28)}_{(+)}$

οπότε $\boxed{\Pi > 0}$